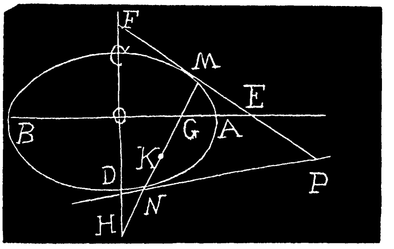
CERTAIN LOCI RELATED TO A CONIC.

By G. B. M. ZERR.

Let M be any point on an ellipse, axes $2a$, $2b$, eccentricity e. O the center of the ellipse, MP the tangent, MN the normal, and MQ the diameter, respectively, at the point M. Let E, F be the points where the tangent MP intersects the axes; G, H the points where the normal intersects the axes. P the pole of MN; K the center of curvature. Let (u, v) be the coördinates of M. Then $(-u, -v)$; $(a^2/u, 0)$; $(0, b^2/v)$; $(e^2u, 0)$; $(0, -a^2e^2v/b^2)$; $(e^2u^3/a^2, -a^2e^2v^3/b^4)$; $[a^2/(e^2u), -b^4/(a^2e^2v)]$; $[u(1-b^2p), v(1-a^2p)]$, where $p=2\left(\frac{a^4v^2+b^4u^2}{a^6v^2+b^6u^2}\right)$, are the coördinates of Q, E, F, G, H, K, P, and N, respectively.



The equation of the perpendicular from O on MP is $y=a^2vx/b^2u$.....(1).

The perpendicular from K on OG is $x=e^2u^2/a^2$.....(2).

The perpendicular from G on OM is $y+\frac{u}{v}(x-e^2u)=0$.....(3).

The perpendicular from K on OH is $y+a^2e^2v^3/b^4=0$.....(4).

The perpendicular from H on OM is $y+a^2e^2v/b^2+ux/v=0$.....(5).

The perpendicular from E on MP is $y-\frac{a^2v}{b^2u}\left(x-\frac{a^2}{u}\right)$.....(6).

The perpendicular from K on MN is $y+\frac{a^2e^2v^3}{b^4}+\frac{b^2u}{a^2v}\left(x-\frac{e^2u^3}{a^2}\right)=0$.....(7)

The perpendicular from F on MP is $y=b^2/v+a^2vx/b^2u$.....(8).

The perpendicular from K on OM is $y+\frac{a^2e^2v^3}{b^4}+\frac{u}{v}\left(x-\frac{e^2u^3}{a^2}\right)=0$..... (9).

The perpendicular from H on NM is $y+\frac{a^2e^2v}{b^2}+\frac{b^2ux}{a^2v}=0$.....(10).

The perpendicular from G on MN is $y+\frac{b^2u}{a^2v}(x-e^2u)=0$.....(11).

The perpendicular from M on OG is $x=u$.....(12).

The perpendicular from M on OH is $y=v$.....(13).

The perpendicular from M on OM is $y-v+\frac{u}{v}(x-u)=0$.....(14).

The line PG is $y(a^4v-a^2e^4u^2v)+b^4ux-b^4e^2u^2=0$.....(15).

The line PH is $y+\frac{a^2e^2v}{b^2}=\frac{(a^4e^4uv^2-b^6u)x}{a^4b^2v}$.....(16).

The line QN is $a^4vy+b^4ux+a^4v^2+b^4u^2=0$.....(17).

The line OK is $y+\frac{a^4v^3x}{b^4u^3}=0$.....(18).

The line through the projections of Q on OG and OH is $uy+vx+uv=0$....(19).

Then we have the following eleven sets of three concurrent lines:

1. (1), (2), (3) meet in $x=e^2u^3/a^2$, $y=u^2v/b^2$.

$\therefore u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=b^2y\sqrt[3]{\frac{e^4}{a^4x^2}}$. Substituting these values of u, v in $b^2u^2+a^2v^2=a^2b^2$, we get for the locus of this point $(a^2x^2+b^2e^4y^2)^3=a^8e^4x^4$.

2. (1), (4), (5) meet in $x=-\frac{e^2uv^2}{b^2}$, $y=-\frac{a^2e^2v^3}{b^4}$.

$\therefore u=-\frac{ax}{e}\sqrt[3]{\frac{ae}{b^2y^2}}$, $v=-b\sqrt[3]{\frac{by}{a^2e^2}}$.

$\therefore$ The locus of this point is $(a^2x^2+b^2y^2)^3=a^4b^4e^4y^4$.

3. (9), (10), (12) meet in $x=u$, $y=-\frac{a^4e^2+b^2u^2-a^2e^2u^2}{a^2v}$.

$\therefore u=x$, $v=-\frac{a^2e^2+b^2x^2-a^2e^2x^2}{a^2y}$.

The locus of this point is $a^2b^2x^2y^2+(a^4e^2+b^2x^2-a^2e^2x^2)^2=a^4b^2y^2$.

4. (9), (11), (13) meet in $x=\frac{a^2b^2e^2-a^2e^2v^2-a^2v^2}{b^2u}$, $y=v$.

$\therefore u=\frac{a^2b^2e^2-a^2e^2y^2-a^2y^2}{b^2x}$, $v=y$.

The locus of this point is $a^2(b^2e^2-e^2y^2-y^2)^2+b^2x^2y^2=b^4x^2$.

5. (2), (8), (14) meet in $x=\frac{e^2u^3}{a^2}$, $y=\frac{a^2b^2+a^2e^2u^2-e^2u^4}{a^2v}$.

$\therefore u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=\left(b^2+ae\sqrt[3]{\frac{ax^2}{e}}-x\sqrt[3]{\frac{a^2x}{e^2}}\right)/y$.

Let $x=z^3$, $a/e=c^3$, then the locus of this point is $b^2c^4y^2z^2+a^2(b^2+acez^2-c^2z^4)^2=a^2b^2y^2$.

6. (4), (6), (14) meet in $x=\dfrac{a^2b^4-a^2b^2e^2v^2+a^2e^2v^4}{b^4u}$, $y=-\dfrac{a^2e^2v^3}{b^4}$.

$$\therefore\ u=\left(a^2-ae\sqrt[3]{\frac{y^2}{ae}}+y\sqrt[3]{\frac{y}{a^2e^2}}\right)/x,\quad v=-b\sqrt[3]{\frac{y}{a^2e^2}}.$$

Let $y=z^3$, $1/ae=c^3$. $\therefore\ u=(a^2-aecz^2+c^2z^4)/x$, $v=-bc^2z$.
The locus of this point is $(a^2-acez^2+c^2z^4)^2+a^2c^4x^2z^2=a^2x^2$.
The loci of 5 and 6 can be found in terms of x and y by expansion.

7. (7), (12), (16) meet in $x=u$, $y=\dfrac{a^2e^2(a^2-u^2)(e^2u^2-a^2)-b^4u^2}{a^4v}$.

$\therefore\ u=x$, $v=(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)/a^4y$.
The locus of this point is $a^6b^2x^2y^2+(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)^2=a^8b^2y^2$.

8. (7), (13), (15) meet in $x=\dfrac{a^2(a^2b^2e^4v^2-a^2e^4v^4-a^2b^2v^2+b^6e^2-b^4e^2v^2)}{b^6u}$

$y=v$. $\therefore\ u=\dfrac{a^2(a^2b^2e^4y^2-a^2e^4y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)}{b^6x}$, $v=y$.

The locus of this point is $a^2(a^2b^2e^4y^2-a^2e^2y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)^2+b^{10}x^2y^2=b^{12}x^2$.

9. (3), (6), (7) meet in $x=\dfrac{b^2e^2u^4+a^4v^2}{a^2b^2u}$, $y=-\dfrac{b^4v+a^2e^2v^3}{b^4}$.

Also, $x=\dfrac{a^2b^4e^2-2a^2b^2e^2v^2+a^2e^2v^4+a^2b^2v^2}{b^4u}$.

10. (5), (7), (8) meet in $x=-\dfrac{a^2u-e^2u^3}{a^2}$, $y=\dfrac{b^6u^2-a^4e^2v^4}{a^2b^4v}$.

Also, $y=\dfrac{b^2u^2-a^4e^2+2a^2e^2u^2-e^2u^4}{a^2v}$.

11. (17), (18), (19) meet in $x=\dfrac{b^4u^3}{a^4v^2-b^4u^2}$, $y=-\dfrac{a^4v^3}{a^4v^2-b^4u^2}$.

Also, $x=\dfrac{b^2u^3}{a^4-(a^2+b^2)u^2}$, $y=\dfrac{a^2v^3}{b^4-(a^2+b^2)v^2}$.

The loci of 9, 10, 11 can be found by solving cubic equations or by means of Calculus. In case 11, it may also be solved as follows:

$$x/y=-\frac{b^4u^3}{a^4v^3} \text{ or } u/v=-\frac{a}{b}\sqrt[3]{\frac{ax}{by}}. \quad \therefore u=-\frac{av}{b}\sqrt[3]{(ax/by)}, \quad v=-\frac{bu}{a}\sqrt[3]{(by/ax)},$$

from values of x, y we get $u+x=\frac{a^2x}{b^2}\sqrt[3]{\frac{b^2y^2}{a^2x^2}}$, $v+y=\frac{b^2y}{a^2}\sqrt[3]{\frac{a^2x^2}{b^2y^2}}$.

These relations also apply to the hyperbola.

ON CERTAIN PROOFS OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

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I.

Until an American text-book on Higher Algebra shall appear, the great majority of American students will probably continue to approach the Fundamental Theorem of Algebra through the well known English texts of Chrystal, Burnside and Panton, or Todhunter. It is therefore to be regretted that these texts, though they aim at considerable rigor in demonstration, fail when it comes to this most important theorem. Yet it is this very theorem where rigor means all, for the mere fact which the theorem embodies, is well known to every student long before he reaches the demonstration in question. It is my pnrpose to point out as briefly as possible certain of these tacit assumptions employed by the several authors, in the hope that if it is not desirable to entirely avoid them, as has been done by Weber in his classic text on algebra, they may at least be explicitly stated as such, in future editions of these and other texts.

II.

Chrystal's* proof is in outline as follows: To prove that one value of z, in general a complex number, can always be found which causes the rational integral function, $f(z)$, to vanish, "we have to show that a value of z can always be found which shall render mod $f(z)$ smaller than any assignable quantity. This will be established if we can show that however small mod $f(z)$ be, provided it be not zero, we can always, by properly altering z, make mod $f(z)$ smaller still." The proof now consists in showing that so long as $f(z)\neq 0$, an increment h of z may be so determined that

$$\text{mod } f(z+h) < \text{mod } f(z);$$

*Chrystal, *Algebra*, I, Chapter XII, §22.